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Radiation and scattering of water waves by rigid bodies

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Schwinger's variational formulation is applied to the radiation of surface waves due to small oscillation of bodies. By means of Haskind's theorem the wave forces on a stationary body due to a plane incident wave are found using only far-field properties. Results for horizontal rectangular and vertical circular cylinders are presented.

1. Introduction

In this paper we wish to exploit further the variational formulation of Schwinger for water-wave problems. This technique, widely used for discontinuous wave guides (Collin 1960), has been successfully applied by Miles (1967), Kelly (1969),

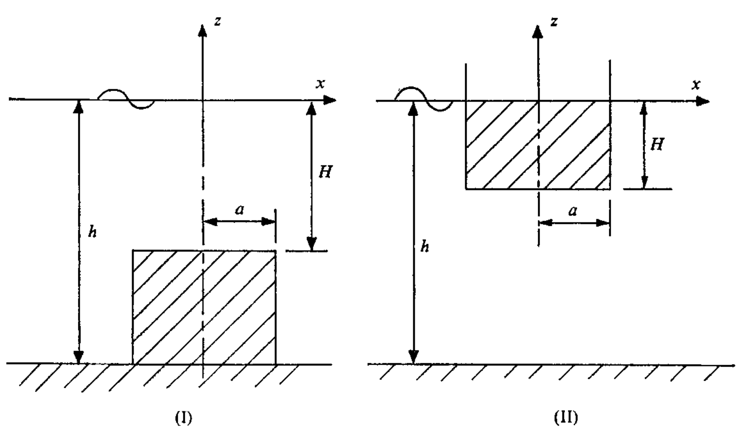

FIGURE 1. Definition sketch.

Mei & Black (1969) and Miles (1971) to water-wave scattering. In all these studies the complex amplitude of the scattered wave in the far field is obtained accurately without striving for equal accuracy for the near field. We shall first show that, by formulating simultaneously the integral equations for the radiation and the scattering problems, the far-field amplitude of the radiated wave can also be calculated variationally. By means of a theorem of Haskind, the information obtained is then used to calculate the force and moment on a fixed body in a plane incident wave.

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We only treat the following two types of rigid bodies in a water of finite depth h: (i) an infinitely long, horizontal cylinder of rectangular cross-section with half width a and (ii) a vertical cylinder of circular cross-section with radius a; the problems considered will be respectively two- and three-dimensional. The height of the body, which can be either protruding from the sea bottom (case I) or partially immersed in the free surface (case II), is assumed to be less than the water depth (see figure 1). These geometries are of ocean-engineering interest as breakwaters, platforms and subsurface storage tanks, etc.

2. Formulation of the radiation problems

In order to unify the analysis for two- and three-dimensional geometries, we use x to denote both the horizontal co-ordinate in a rectangular system (x, z) for the former, and the radial co-ordinate in a cylindrical polar system (x, θ, z) for the latter.

Consider a body oscillating in a plane (x, z for two-dimensional and $\theta = 0$ and π for three-dimensional), with frequency ω. The velocity potential $\phi e^{i\omega t}$ satisfies (Wehausen & Laitone 1960)

$$\left.\begin{aligned} &\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2}\right)\phi(x,z) = 0 \quad \text{(2-D)} \\ &\left(\frac{\partial^2}{\partial x^2}+\frac{1}{x}\frac{\partial}{\partial x}+\frac{1}{x^2}\frac{\partial^2}{\partial \theta^2}+\frac{\partial^2}{\partial z^2}\right)\phi(x,\theta,z) = 0 \quad \text{(3-D)}\end{aligned}\right\} \quad \text{(in the fluid)}, \tag{2.1 a, b}$$

$$\frac{\partial \phi}{\partial z} - \sigma\phi = 0, \quad z = 0 \quad \text{(free surface)}, \tag{2.2}$$

$$\frac{\partial \phi}{\partial z} = 0, \quad z = -h \quad \text{(sea bottom)}, \tag{2.3}$$

with $\sigma = \omega^2/g$. The potential must also represent an outgoing wave at large x. Let $V_j = \mathscr{V}$, $\mathscr{U}$ and Ω, $j = z, x, \lambda$, be the velocity amplitude of the forced heave, sway and roll about a horizontal axis through $z = c$, respectively. We must further require on the body that

$$\left.\begin{aligned} &\partial\phi/dz = \mathscr{V} + \Omega x \quad \text{(2-D)} \\ &\partial\phi/\partial z = (\mathscr{V} + \Omega x)\cos\theta \quad \text{(3-D)}\end{aligned}\right\} \quad \left.\begin{aligned} &(z = -H), \\ &(|x| < a),\end{aligned}\right\} \tag{2.4 a, b}$$

on the top [bottom] for case I [II], and

$$\left.\begin{aligned} &\partial\phi/\partial x = \mathscr{U} + \Omega(c-z) \quad \text{(2-D)} \\ &\partial\phi/\partial x = [\mathscr{U} + \Omega(c-z)]\cos\theta \quad \text{(3-D)}\end{aligned}\right\} \quad \left.\begin{aligned} &(|x| = a), \\ &(z \in \mathscr{Z}^W),\end{aligned}\right\} \tag{2.5 a, b}$$

where $\mathscr{Z}^W$ denotes the vertical range of the side wall (hence the superscript W).

It is clear that, in the two-dimensional problems, the potential induced by the forced heave (sway and roll) is even (odd) in x; hence we decompose the potential as follows:

$$\phi(x,z) = \phi_0(x,z) + \phi_1(x,z), \tag{2.6}$$

where the subscripts $m = 0, 1$ refer to even and odd modes respectively. In this way only the region $x > 0$ needs to be considered. Similarly, the decomposition

$$\phi(x,\theta,z) = \phi_0(x,z)+\phi_1(x,z)\cos\theta \tag{2.7}$$

can be made for the three-dimensional problems in view of the angular dependence of the boundary conditions (2.4*b*) and (2.5*b*). Here the subscripts m refer to the angular modes. For both classes the boundary conditions on the body are decomposed to:

$$\partial\phi_m/\partial x = U_m^W(z) \quad (x = a, \quad z\in\mathscr{Z}^W), \tag{2.8a}$$

with
$$U_0^W(z) = 0, \quad U_1^W = \mathscr{U}+\Omega(c-z), \tag{2.8b}$$

and
$$\partial\phi_m/\partial z = V_m(x) \quad (0 < x < a, \quad z = -H), \tag{2.9a}$$

with
$$V_0 = \mathscr{V}, \quad V_1 = \Omega x. \tag{2.9b}$$

Other conditions on ϕ apply also on ϕ_m $(m = 0, 1)$.

In problems of scattering by a fixed body, the formulation is quite similar; for details we refer to Mei & Black (1969) for the two-dimensional cases and to Miles & Gilbert (1968) for the three-dimensional cases.

3. Reduction to integral equations

We shall first transform the boundary-value problems to integral equations. Let the following symbols be introduced for various domains of z:

$$\left.\begin{aligned} &\mathscr{Z}: \begin{cases} -H < z < 0 & \text{(I)}, \\ -h < z < -H & \text{(II)}, \end{cases} \\ &z = \mathscr{Z}+\mathscr{Z}^W: \quad (-h < z < 0). \end{aligned}\right\} \tag{3.1}$$

In terms of eigenfunctions, the potential can be expressed as

$$\phi_m = \sum_{n=0}^{\infty} b_{mn} f_n(z)\,\psi_{mn}(x) \quad (x > a,\ z\in z), \tag{3.2}$$

for the outer region and

$$\Phi_m = \Phi_m^p(x,z)+\sum_{n=0}^{\infty} B_{mn}F_n(z)\Psi_{mn}(x) \quad (0 < x < a,\ z\in\mathscr{Z}), \tag{3.3}$$

for the inner region, which is distinguished by capital symbols. In the preceding equations, the sets $\{f_n\}$ and $\{F_n\}$, $n = 0, 1, 2, \ldots$, represent the vertical eigenfunctions which are orthonormal in z and $\mathscr{Z}$ respectively. The functions ψ_{mn} and Ψ_{mn} represent the horizontal eigensolutions with ψ_{m0} being the propagating mode behaving as outgoing waves at large x and $\psi_{mn} (n \geqq 1)$ being the evanescent modes exponentially decaying at large x. Φ_m^p are the particular solutions for the inner region, harmonic and satisfying (2.2) [(2.3)] for case I [II] and the inhomogeneous condition at $z = -H$. Reference is made to the appendix for their explicit expressions.

Matching the horizontal velocity at the surface $x = a$, the Fourier coefficients can be expressed in terms of the unknown $\partial\Phi_m/\partial x$ at $x = a$. Introducing

$$\left.\frac{\partial\Phi_m}{\partial x}\right|_{x=a} = U_m^p(z)+W_m(z), \quad U_m^p(z) \equiv \left.\frac{\partial\Phi_m^p}{\partial x}\right|_{x=a}, \tag{3.4}$$

and denoting $(\partial\alpha/\partial x)|_{x=a}$ by $\alpha'(a)$, we have in particular,

$$b_{m0} = \frac{1}{\psi'_{m0}(a)}[\langle U^p_m, f_0\rangle + \langle W_m, f_0\rangle + \langle U^W_m, f_0\rangle], \tag{3.5}†$$

which is directly related to the amplitude of the radiated wave at large x. Matching further the potentials at $x = a$, $z \in \mathscr{X}$, an integral equation is obtained:

$$\mathscr{L}_m W_m \equiv \int_{\mathscr{X}} d\zeta\, G_m(z|\zeta)\, W_m(\zeta) = \gamma_m(z) \quad (z \in \mathscr{X}); \tag{3.6a}$$

where
$$G_m(z|\zeta) = \sum_{n=0}^{\infty} \frac{\psi_{mn}(a)}{\psi'_{mn}(a)} f_n(z) f_n(\zeta) - \sum_{n=0}^{\infty} \frac{\Psi_{mn}(a)}{\Psi'_{mn}(a)} F_n(z)\, F_n(\zeta), \tag{3.6b}$$

and
$$\gamma_m(z) = \Phi^p_m(z) - \sum_{n=0}^{\infty} \frac{\psi_{mn}(a)}{\psi'_{mn}(a)} f_n(z)\, [\langle U^p_m, f_n\rangle + \langle U^W_m, f_n\rangle]. \tag{3.6c}$$

We note that the kernel G_m is symmetric in z and ζ. The above integral equation replaces the original boundary-value problem for $m = (0, 1)$, case I and $m = 1$, case II. The heaving of a surface dock ($m = 0$, case II) needs some care since $\Psi'_{00}(a) = 0$. As can be easily shown, (3.6a) ought to be modified to

$$\int_{\mathscr{X}} d\zeta\, G_0(z|\zeta)\, W_0(\zeta) = \gamma_0(z) + B_{00} F_0 \quad \text{(II)}, \tag{3.6d}$$

where G_0 is obtained from (3.6b) by deleting the term $n = 0$ from the second series. A further constraint from mass conservation,

$$\int_{\mathscr{X}} dz U_0(z) = \int_{\mathscr{X}} dz\, W_0(z) = 0 \tag{3.6e}$$

should be added to render the problem determinate (Garrett 1971).

In order to achieve a stationary form for b_{m0} (hence $\langle W_m, f_0\rangle$), an adjoint integral equation with the same kernel and with f_0 on the right-hand side is needed (Jones 1963, p. 270). Take the two-dimensional cases first. For $m = (0, 1)$, this is provided by the (even, odd) scattering by the same body, now fixed, of two normally incident waves at equal amplitude and (same, opposite) phase.‡ With slight modifications from the treatment of Mei & Black (1969), we obtain the governing integral equation for the corresponding horizontal velocity U^s_m at $x = a$, $z \in \mathscr{X}$:

$$\mathscr{L}_m U^s_m = \int_{\mathscr{X}} d\zeta\, G_m(z|\zeta)\, U^s_m(\zeta) = C_m d_0 f_0(z), \tag{3.7}$$

for $m = 0$ and 1 (case I) and $m = 1$ (case II) where C_m is some constant unimportant for the present discussion and d_0 is the amplitude of the incident-wave potential, defined by

$$\phi^i = d_0 f_0(z)\, e^{-ikx}. \tag{3.8}$$

Similar to the discussions leading to (3.6c) and (3.6d), for the even scattering of a surface dock ($m = 0$, case II) we must replace (3.7) by

$$\mathscr{L}_0 U^s_0 = C_0 d_0 f_0 + D_{00} F_0, \tag{3.9a}$$

† The range of integration for all inner-products $\langle u, v\rangle \equiv \int uv\, dz$ will be understood as the interval in which both u and v are defined, e.g. $\mathscr{X}^W$ for $\langle U^W_m, f_n\rangle$ and $\mathscr{X}$ for $\langle U^p_m, f_n\rangle$.

‡ This fact was utilized by Levine (1958) in a related problem.

with the kernel redefined in the same way. The following additional constraint is also imposed

$$\int_{\mathscr{X}} dz\, U_0^s(z) = 0 \quad \text{(II)}. \tag{3.9b}$$

The three-dimensional problems are treated similarly. In polar co-ordinates the incident wave from $\theta = \pi$ (same as (3.8)) is represented by:

$$\phi^i = d_0 f_0 e^{-ikx\cos\theta} = d_0 f_0 \sum_m \epsilon_m(-i)^m J_m(kx)\cos m\theta \quad (\epsilon_0 = 1, \epsilon_m = 2; m = 1, 2, \ldots). \tag{3.10}$$

In view of the above expansion, we may express the scattering potential $\phi^s = \phi - \phi^i$ as a series of $\cos m\theta$. Each angular mode $m = 0, 1, 2, 3, \ldots$ leads to an integral equation of precisely the form of (3.7) with similar modifications for $m = 0$, case II (isotropic scattering by a surface dock).

4. Variational calculation for the far field

As mentioned previously, the amplitude of the radiated wave at large x is determined by $S_m^r = \langle W_m, f_0\rangle$. From parallel analysis it can be shown that the amplitude of the scattering potential at large x is determined by $S_m^s = \langle U_m^s, f_0\rangle$ (Miles & Gilbert 1968). By virtue of the symmetry of the integral operators $\mathscr{L}_m$, these functionals are stationary if expressed in the following forms of Schwinger:

$$S_m^r = \frac{\langle W_m, f_0\rangle \langle U_m^s, \gamma_m\rangle}{\langle W_m, \mathscr{L}_m U_m^s\rangle}, \tag{4.1}$$

and

$$S_m^s = \frac{\langle U_m^s, f_0\rangle^2}{\langle U_m^s, \mathscr{L}_m U_m^s\rangle}. \tag{4.2}$$

In general, S_m^r is stationary with respect to independent variations of W_m and U_m^s (Jones 1963, p. 270); except for $m = 0$, case II, the stationarity is subjected to the constraint of (3.6*e*) and (3.9*b*) with the coefficients B_{00} and D_{00} in (3.6*d*) and (3.9*a*) being the Lagrange multipliers (Miles 1971). It is interesting that while they are crucial when force calculations are made by direct integration of the pressure on the body (Garrett 1971), these coefficients do not appear explicitly in the present variational formulation. The stationarity of S_m^s, in the form (4.2), may be regarded as a special case of (4.1) by taking $\gamma_m = f_0$, and $W_m = U_m^s$.

We now apply the Rayleigh–Ritz procedure. Expressing W_m and U_m^s as N-term truncated series of orthonormal functions $\{F_i\}$,

$$W_m(z) = \sum_{i=0}^{N} \mu_i F_i, \quad U_m^s(z) = \sum_{j=0}^{N} \nu_i F_i \tag{4.3}$$

the conditions $(\partial/\partial\mu_i, \partial/\partial\nu_j)\, S_m^r = 0$ $(i, j = 0, 1, 2, \ldots, N)$, then lead to the following explicit result without having to solve for any coefficient μ_i or ν_j (Collin 1960, p. 335):

$$S_m^r = \begin{vmatrix} \cdots & Q_j & \cdots \\ & \vdots & \\ \cdots & \left(\dfrac{g_{ij}}{P_i} - \dfrac{g_{0j}}{P_0}\right) & \cdots \\ & \vdots & \\ \cdots & \cdots & \cdots \end{vmatrix} \div \begin{vmatrix} \cdots & \dfrac{g_{0j}}{P_0} & \cdots \\ & \vdots & \\ \cdots & \left(\dfrac{g_{ij}}{P_i} - \dfrac{g_{0j}}{P_0}\right) & \cdots \\ & \vdots & \\ \cdots & \cdots & \cdots \end{vmatrix}, \tag{4.4}$$

where $$g_{ij} = \langle F_i, \mathscr{L}_m F_j \rangle, \quad P_i = \langle F_i, f_0 \rangle, \quad Q_j = \langle F_j, \gamma_m \rangle.$$

For $m = 0$, case II, a slight modification is needed, as the terms $i = 0$ and $j = 0$ must be omitted from (4.3) as required by (3.6e) and (3.9b).

With the amplitude of the potential thus found, the corresponding amplitude of the wave height A^r_m defined by

$$\left.\begin{aligned} &\eta^r \sim A^r_m\, e^{-ikx} \quad \text{(2-D)} \\ &\eta^r \sim A^r_m \cos m\theta\, (a/x)^{\frac{1}{2}}\, e^{-ikx} \quad \text{(3-D)} \end{aligned}\right\} \quad (kx \gg 1), \tag{4.5a, b}$$

may be inferred from the linearized Bernoulli equation $\eta = -(i\omega/g)\,\phi|_{z=0}$:

$$A^r_m = -\frac{i\omega}{g} b_{m0} f_0(0) \begin{Bmatrix} e^{ika} & \text{(2-D)}, \\ (2/\pi ka)^{\frac{1}{2}}\, e^{i(\frac{1}{4}\pi + \frac{1}{2}m\pi)} & \text{(3-D)} \end{Bmatrix}, \tag{4.6a, b}$$

[cf. (A 3a) and (A 4a)].

It may be shown that the average rate of energy flux due to oscillating bodies is proportional to the square of the radiated wave amplitude in the far field, from which the damping coefficient, a quantity of engineering importance, can be obtained (Newman 1962). Because of the simple relationship we do not pursue the matter here.

5. Forces and moment on a stationary body

A theorem due to Haskind (see Newman 1962) enables one to calculate the wave forces by using only the potential of the incident wave and the far-field potential of the radiated wave generated by an oscillating body. Let $\hat{\phi}^r_j \,.\, e^{i\omega t}$ be the radiation potential corresponding to the oscillating mode j ($j = z, x, \lambda$), normalized for unit velocity amplitude (i.e. $V_j = 1$). The hydrodynamic forces and moment due to a plane incident wave on a stationary body of the same geometry are given by:

$$F_j = -i\rho\omega \iint_{\Sigma_\infty} \left(\phi^i \frac{\partial \hat{\phi}^r_j}{\partial n} - \hat{\phi}^r_j \frac{\partial \phi^i}{\partial n} \right) d\Sigma \quad (j = z, x, \lambda), \tag{5.1}$$

$$\begin{bmatrix} F_z \\ F_x \\ F_\lambda \end{bmatrix} .\, e^{i\omega t} = \text{total} \begin{cases} \text{vertical force,} \\ \text{horizontal force,} \\ \text{moment about } z = c, \end{cases}$$

where Σ_∞ is a vertical cylindrical control surface at large x. By singling out a particular mode of oscillation, the amplitude of the radiation potential $(b_{m0})_j$ can be easily obtained from b_{m0}. It then follows after simple calculations that

$$F_j = 2\rho\omega k\, d_0 \frac{(b_{m0})_j}{V_j} \quad \text{(per unit length)} \quad \begin{Bmatrix} m = 0 : j = z \\ m = 1 : j = x, \lambda \end{Bmatrix}, \tag{5.2a}$$

for unit length of the rectangular cylinder (Newman 1962), and

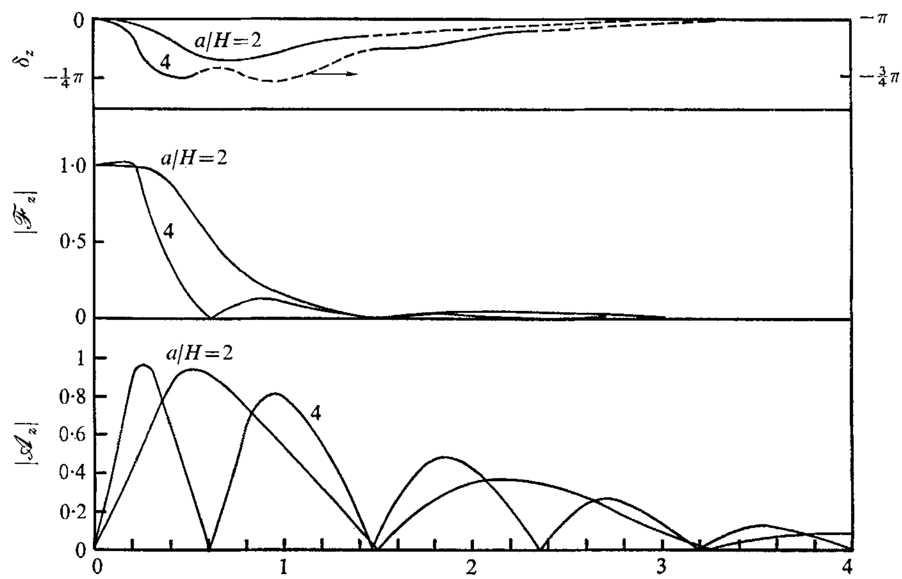
$$F_j = 4(-i)^m \rho\omega\, d_0 (b_{m0})/V_j \quad \text{(3-D)} \tag{5.2b}$$

for the circular cylinder. The coefficient d_0 is the amplitude of the incident-wave potential, as defined by (3.8) and (3.10).

6. Numerical results

Radiated-wave amplitudes A_j^r (magnitude and phase) have been calculated when the body executes only one mode of oscillation. Denoting the maximum body displacement by $(z_0, x_0, \lambda_0) = -(1/\omega)(\mathscr{V}, \mathscr{U}, \Omega)$, we normalize A_j^r and define the phase as follows:

$$\mathscr{A}_j = |\mathscr{A}_j| \, e^{i\delta_j} = \left(\frac{A_z^r}{z_0}, \frac{A_x^r}{x_0}, \frac{A_\lambda^r}{\lambda_0 a}\right). \tag{6.1}†$$



(*a*)

FIGURE 2. Horizontal rectangular cylinder on bottom ($h/H = 2$). Radiated-wave amplitude (magnitude and phase) due to body oscillation, and wave forces due to scattering: (*a*) vertical, (*b*) horizontal, (*c*) rotational about ($x = 0$, $z = -h$). For thin plate ($a/H = 0$) replace a by $h - H$ in equations (6.1) and (6.2).

The vertical and horizontal forces and the moment about $z = c$ are normalized with respect to a characteristic hydrostatic force due to the incident wave:

$$\mathscr{F}_j = |\mathscr{F}_j| \, e^{i\beta_j} = \begin{Bmatrix} F_{z,x}/2\rho g a A, & F_\lambda/2\rho g a^2 A & \text{(2-D)} \\ F_{z,x}/\pi\rho g a A h_0, & F_\lambda/\pi \rho g a^2 A h_0 & \text{(3-D)} \end{Bmatrix}, \tag{6.2}†$$

where h_0 is the net height of the body, i.e. $h - H$ for case I and H for case II, and A is the incident wave amplitude $A = -(i\omega/g)\, d_0 f_0(0)$.

The phase angles δ_j and β_j are simply related as may be shown from (4.6), (5.2), (6.1) and (6.2)

$$\delta_j = \beta_j \quad (j = z, x, \lambda) \quad \text{(2-D)};$$

$$\beta_z = \delta_z - \tfrac{1}{4}\pi, \quad \beta_{x,\lambda} = \delta_{x,\lambda} + \tfrac{3}{4}\pi \quad \text{(3-D)}. \tag{6.3}$$

† See figure captions for exceptions.

Computational aspects are similar to those discussed in Mei & Black (1969). Sample results are shown in figures 2(*a*), (*b*), (*c*) for rectangular cylinders (2-D) and figures 3(*a*), (*b*), (*c*) for circular cylinders (3-D), all protruding from the sea bottom (case I). The vertical force is seen to approach the correct long-wave limit at $kH \to 0$ or $ka \to 0$, i.e. the additional hydrostatic force due to the free surface elevation $F_z = 2\rho g a A$ (2-D), or $\pi \rho g a^2 A$ (3-D). We note

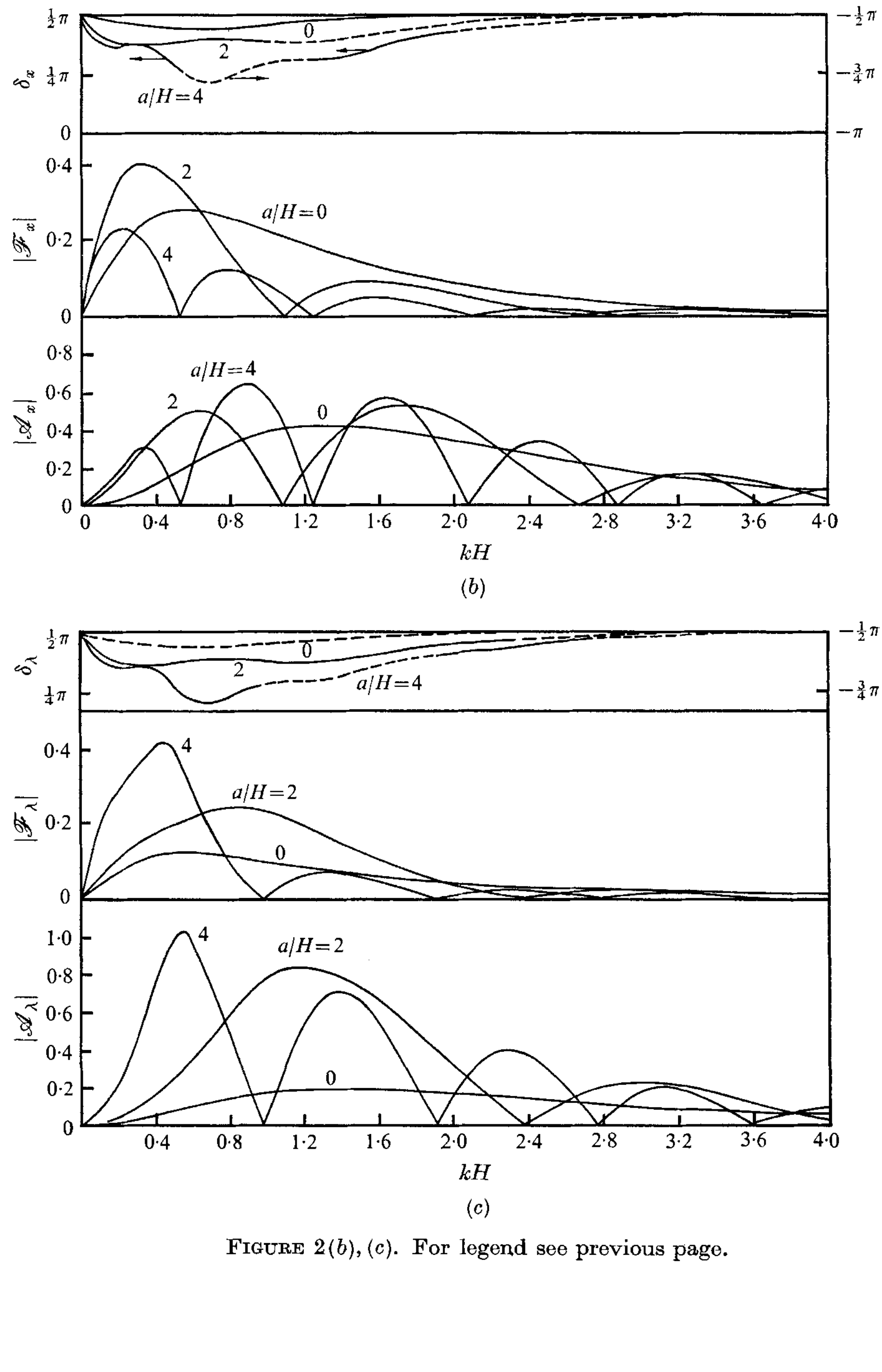


FIGURE 2(*b*), (*c*). For legend see previous page.

the occurrence of nodes where the complex radiation amplitude and the force coefficients change sign; the phase angles defined here therefore change by π abruptly. We present the variation as continuous curves by using a dual co-ordinate system, i.e. read the ordinate on the left for solid curves and on the right for dashed curves. This change of phase is brought about by the interference due to contributions from various parts of the body. Figures 4 (*a*), (*b*), (*c*) are for a

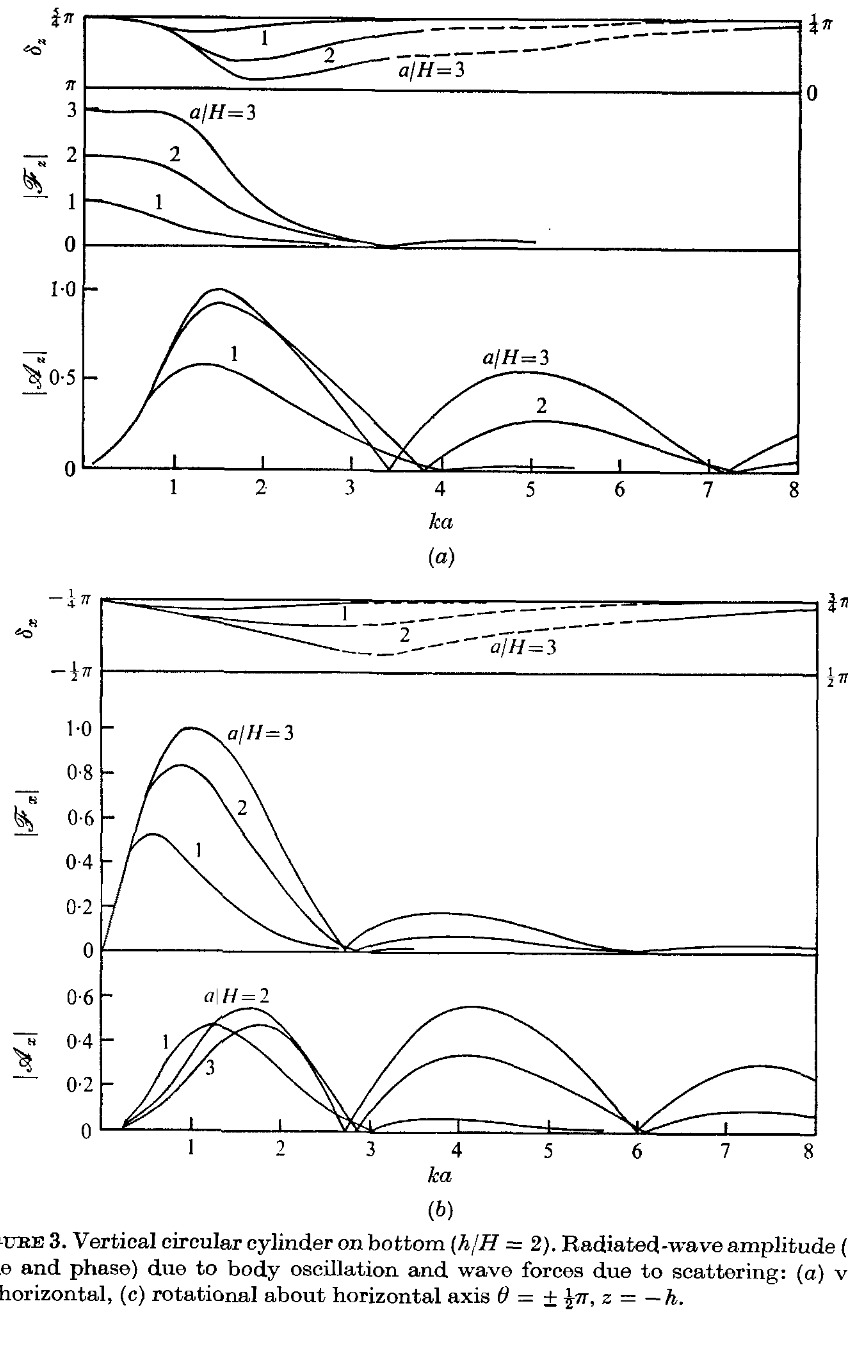


FIGURE 3. Vertical circular cylinder on bottom ($h/H = 2$). Radiated-wave amplitude (magnitude and phase) due to body oscillation and wave forces due to scattering: (*a*) vertical, (*b*) horizontal, (*c*) rotational about horizontal axis $\theta = \pm\frac{1}{2}\pi$, $z = -h$.

horizontal rectangular cylinder in the free surface (case II). We omit the corresponding results for the circular dock which has been recalculated independently by Garrett (1971) to rectify earlier errors of Miles & Gilbert (1968); suffice it to mention that for the same geometrical parameters the agreement of our results is excellent. Comparison with the limiting case of a cylinder extending the entire depth (i.e a sea wall in 2-D, or a pile in 3-D) for which exact formulas are known

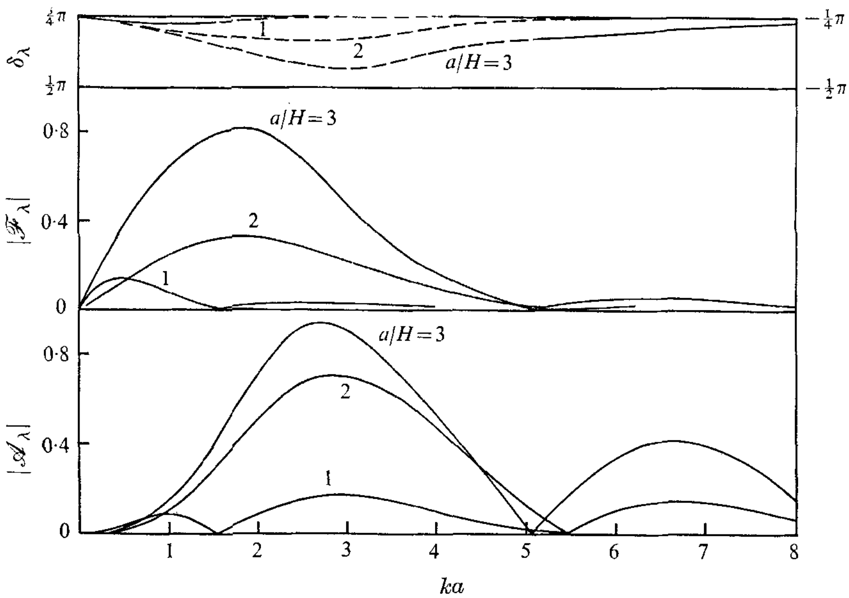


FIGURE 3(*c*). For legend see previous page.

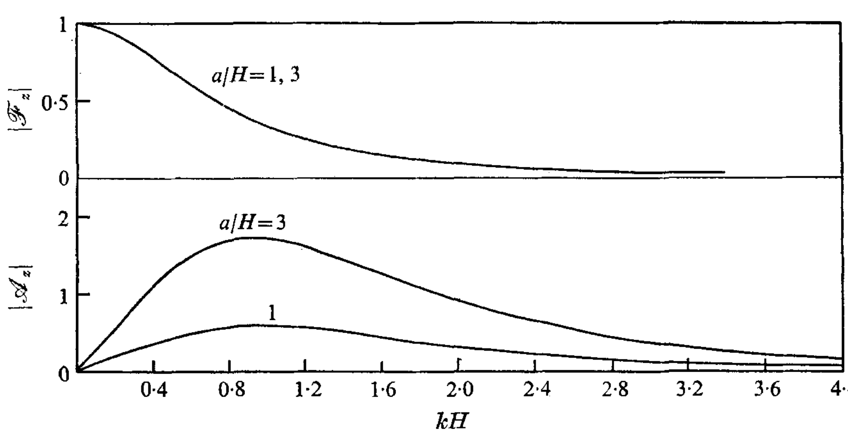
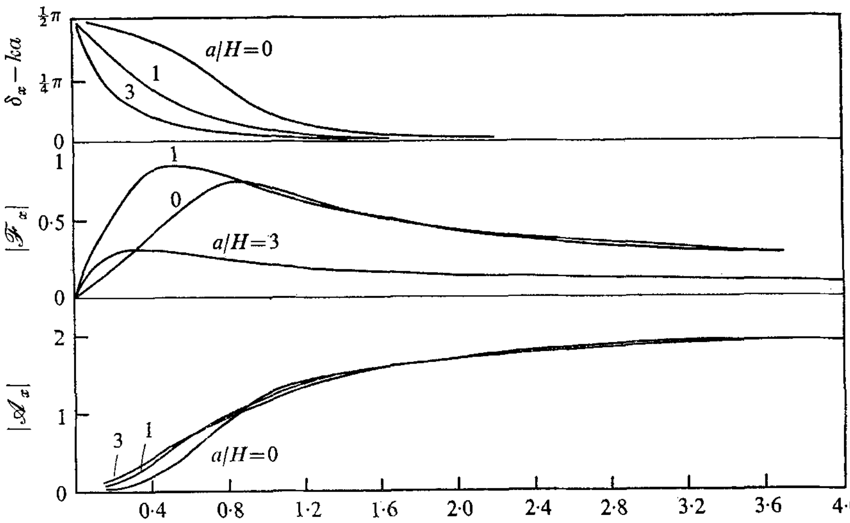


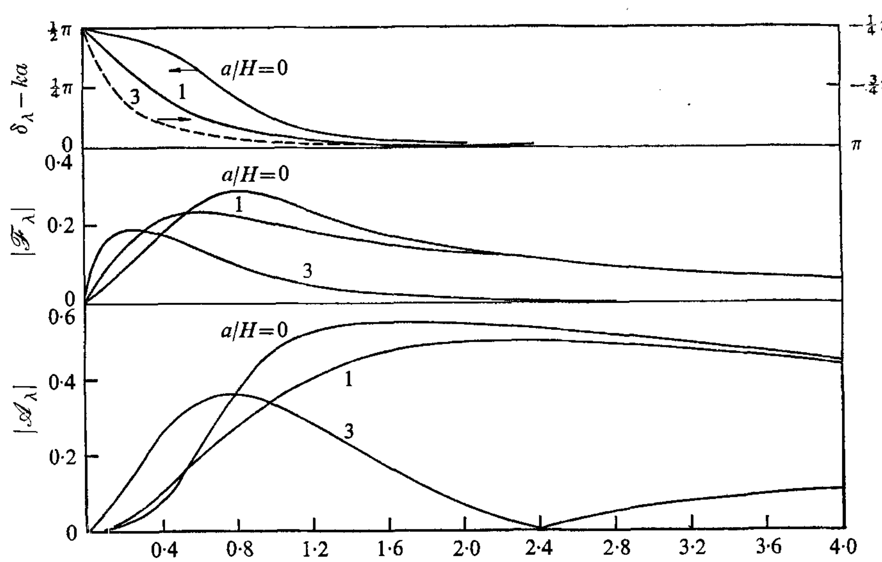
FIGURE 4. Horizontal rectangular cylinder in free surface ($h/H = 2$). Radiated wave amplitude (magnitude and phase) due to body oscillation and wave forces due to scattering: (*a*) vertical ($\delta_z = ka + \pi$), (*b*) horizontal, (*c*) rotational about ($x = 0$, $z = 0$). For thin plate ($a/H = 0$) replace a by H in equations (6.1) and (6.2).

have also been made but not shown here. For case II the known limits are approached smoothly as $H \to h$. But for case I when the depth of the top H is reduced, interference intensifies so that the number of nodes increases; of course due to the very shallow depth over the top of the body, the deductions from a linearized theory must be received with caution.

We also present the total scattering cross-section, defined in Miles & Gilbert (1968), for both bottom and surface cylinders of circular plan form, figures 5(*a*)



(*b*)



(*c*)

FIGURE 4(*b*), (*c*). For legend see previous page.

and (*b*); for the latter case the original results of Miles & Gilbert were in error (Miles 1971).

For further details and more extensive numerical results reference is made to Black & Mei (1970).

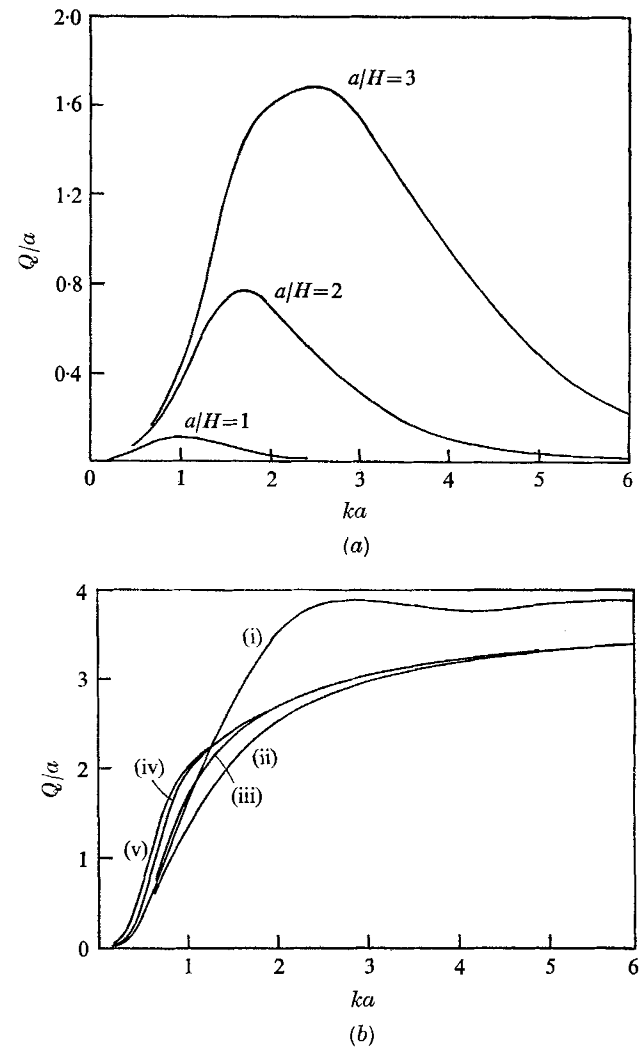


FIGURE 5. Total scattering cross-section of vertical circular cylinder: (*a*) on bottom, $h/H = 2$; (*b*) in free surface. $R/h = \frac{1}{4}$ values of H/h: (i), 0; (ii), $\frac{1}{8}$; (iii), $\frac{1}{4}$; (iv), $\frac{1}{2}$; (v), 1.

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Appendix. Details of particular and eigensolutions

These solutions can all be obtained by separation of variables (see Miles 1967; Miles & Gilbert 1968; Mei & Black 1969).

Vertical eigenfunctions

The outer region: $x > a$, $z \in x$.

$$f_n = \frac{\sqrt{2}\cos k_n(z+h)}{(h-(1/\sigma)\sin^2 k_n h)^{\frac{1}{2}}}, \quad k_n \tan k_n h = -\sigma, \tag{A 1}$$

where $k_0 \equiv ik$; k, k_n $(n = 1, 2, \ldots)$ are real and positive.

The inner region $(0 < x < a)$. For case (I), expressions for $F_n(z)$ can be obtained from (A 1) by just changing (f, k, h) to (F, K, H); for case (II), they are

$$F_0 = (h-H)^{-\frac{1}{2}}, \quad F_n = \frac{\sqrt{2}\cos K_n(z+H)}{(h-H)^{\frac{1}{2}}},$$
$$K_n = n\pi/(h-H) \quad (n = 1, 2, 3, \ldots). \tag{A 2}$$

Equations (A 1) and (A 2) are valid for both two- and three-dimensional problems treated here.

Horizontal eigenfunctions

Two-dimensions.

$$\left.\begin{aligned} &\psi_{mn}(x) = e^{-k_n(x-a)}, \quad x > a \quad (m = 0, 1;\ n = 0, 1, 2, \ldots) \\ &\Psi_{00}(x) = \begin{pmatrix}\cos Kx \\ 1\end{pmatrix}, \quad \Psi_{0n}(x) = \cosh K_n x \begin{cases}(\mathrm{I}), \\ (\mathrm{II}),\end{cases} \\ &\Psi_{10}(x) = \begin{pmatrix}\sin Kx \\ x\end{pmatrix}, \quad \Psi_{1n}(x) = \sinh K_n x \begin{cases}(\mathrm{I}), \\ (\mathrm{II}),\end{cases} \\ &\qquad (0 < x < a, \quad n = 1, 2, 3, \ldots). \end{aligned}\right\} \tag{A 3a–c}$$

Three-dimensions.

$$\left.\begin{aligned} &\psi_{m0} = H_m^{(2)}(kx), \quad \psi_{mn} = K_m(k_n x) \quad (n = 1, 2, \ldots), \\ &\Psi_{m0} = \begin{pmatrix}J_m(Kx) \\ x^m\end{pmatrix}, \quad \Psi_{mn} = I_n(K_n x) \quad (n = 1, 2, \ldots) \begin{cases}(\mathrm{I}), \\ (\mathrm{II}),\end{cases} \end{aligned}\right\} \tag{A 4a, b}$$

where $H_m^{(2)}$, K_m, J_m and I_m are standard Bessel functions.

The particular solutions are

$$\left.\begin{aligned} &\Phi_0^p = \begin{cases}\mathscr{V}(z+(1/\sigma)) & (\mathrm{I}), \\ \mathscr{V}[(z+h)^2 - x^2]/2(h-H) & (\mathrm{II}),\end{cases} \\ &\Phi_1^p = \Omega x(z+(1/\sigma)) \qquad (\mathrm{I}), \end{aligned}\right\} \tag{A 5a–c}$$

for both two- and three-dimensions, and

$$\left.\begin{aligned} &\Phi_1^p = \Omega x[(z+h)^2 - \tfrac{1}{3}x^2]/2(h-H) \quad (\text{2-D}), \\ &\Phi_1^p = \Omega x[(z+h)^2 - \tfrac{1}{4}x^2]/2(h-H) \quad (\text{3-D}), \end{aligned}\right\} \tag{A 5d, e}$$

for case II.

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